On LQ-Control of Magnetic Bearing

Yuri N. Zhuravlyov

***Abstract*--Control theory gives very few examples of control systems for which the closed-form solution to the Linear-Quadratic (LQ) optimization problem exists. This paper describes two such systems of 2nd and 4th order concerning magnetic bearings and gives the closed-form solutions to the LQ-problems. The controller obtained provides the LQ-optimal bearing forces and minimizes copper losses in coils. The closed-loop system has a variable structure. Stability of the system is analyzed by using the Van der Pol method. Theoretical results are verified by simulations and experiments. The problems of controller simplification are also discussed.**

I. Introduction

Due to their having no mechanical contact and requiring no lubrication, Active Magnetic Bearings (AMBs) are often used for suspending high speed rotors subjected to gyroscopic and disturbance forces. The stable suspension of the rotor is provided by magnetic attractive forces exerted by controlled electromagnets. Physical limitations imposed on bearing forces, currents and voltages lead quite naturally to the formulation of a control strategy as an optimization problem where the goal is to minimize values of control variables. However, AMBs must also provide a specified accuracy in the rotor positioning, with higher positioning accuracies requiring greater values of control variables. The optimal compromise can be obtained by minimizing a perfomance index which would include both of the above requirements. Control theory offers a great number of perfomance indices and methods of optimization. As it follows from practical applications, one of the most suitable approaches for AMB control seems to be that of Linear-Quadratic (LQ) optimal control [1] – [3]. We shall use the following interpretation of this method. Consider a system modeled by a controllable and observable state-space description

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ q(t) &= C\,x(t) \end{aligned} \tag{1}$$

where *x(t)* is the n_{sys}-vector of state variables, *q(t)* the *m*-vector of output variables, *u(t)* the *m*-vector of input variables, and *A*,*B*,*C* are all constant matrices of appropriate

Pskov Polytechnic Institute,

L. Tolstogo st., 4, Pskov, 180680, Russia

Tel: +7 811-22 35101

Fax: +7 811-22 29732

E-mail: pec@fitacom.ru

dimensions. It is required to find the control $u=u^0$ which brings the system (1) from an arbitrary initial state to the zero state while minimizing the quadratic integral perfomance index

$$\int_0^\infty \left[q^T(t)\, q(t) + \rho\, u^T(t)\, u(t) \right] dt \tag{2}$$

where ρ is a positive weighting scalar. As evident from (2), we consider the case where all the output variables are weighted equally, as are the input variables. The optimal control is known to be given by a linear transformation of the state vector

$$u^0(t) = -\rho^{-1} B^T P x(t) \tag{3}$$

where the $n_{sys} \times n_{sys}$ symmetric matrix P is the only positive definite solution to the algebraic matrix Riccati equation

$$C^T C + A^T P + PA - \rho^{-1} PBB^T P = 0 \tag{4}$$

It should be mentioned that $x(t)$, $q(t)$ and $u(t)$ can also be taken as complex vectors, with a complex system matrix A and a Hermitian Riccati matrix P, and with the "T" denoting the conjugate transposition. The complex variable notation is often used in dynamic analysis and control of rotational systems (see, for example, [4], [5], and references therein). Since the order of a system treated in the complex state-space approach is half of that in the real approach, the optimal controller design is far simpler and more comprehensive.

The main difficulty of the task lies in the solution of Riccati Eq.(4) which represents a set of $n_{sys}(n_{sys}+1)/2$ non-linear algebraic equations. Control theory gives very few examples of control systems for which the analytical solution to this Riccati equation exists. Therefore, the optimal AMB control design is usually based either on a numerical solution of (4) or, most often, on the pole placement method [6] combined with use of the asymptotic properties of the optimal closed-loop poles for $\rho \to 0$ and $\rho \to \infty$, [3].

The principal theme of this paper is to show that there are at least two kinds of controller design problems regarding AMBs for which there exist an analytical solution to the Riccati equation and, therefore, closed-form solutions to the LQ-optimization problem.

One of the problems considered is that of designing a typical linear One-Degree-Of-Freedom (1-DOF, or second-order) current control system. The LQ-design for this system is well-studied for the limiting cases $\rho \to \infty$ ("expensive" control) and $\rho \to 0$ ("cheap" control), [6]. In this paper the LQ-problem is solved for an arbitrary value of ρ. For convenience of designers, parameter ρ is expressed in terms of the undamped natural frequency of the closed-loop system to provide a more physically intuitive interpretation.

The other design problem considered is that of designing a common 5-DOF voltage control system for suspending a rigid gyroscopic rotor which has linear mechanical and non-linear electromagnetic subsystems. The mechanical subsystem consists of three 1-DOF (or second-order) systems which describe translational motions of the rotor, and one 2-DOF (or fourth-order) gyroscopic system which describes rotational motions of the rotor. In this paper the LQ-optimal control forces and moments are found. It should be noted that in the case of the 2-DOF system a fourth-order real (or a second-order complex) Riccati equation is analytically solved.

The optimal control currents and voltages are determined from the electromagnetic subsystem in such a way that they produce the optimal forces and moments and, at the same time, minimize copper losses in the coils. Such an approach leads to the switching drive control or the so-called external linearization [6]. Because of errors in tracking the control currents, the problem of stability of the closed-loop system arises. In this paper the problem is treated theoretically by using the Van der Pol method, and by simulations and experiments.

The other problem discussed in the paper is the simplification of the controller obtained. This controller is switching, multi-coupled and speed-dependent, i.e. relatively complicated; controller simplification might be considered advisable. However, it is shown that there are applications where simplifying the controller leads to a significant deterioration of the system perfomance.

II. One --DOF AMB System

The model of the system is shown in Fig.1. The system incorporates a ferromagnetic body of mass m and two counteracting electromagnets with coil currents i_1 and i_2 . The body can move only in the y-direction. The goal is to stabilize the body at the operating point given by the bias currents $i_1=i_2=i_0$ and the reference position y=0 at which both the air gaps have the nominal value δ. Forming the currents as $i_1=i_0+i$, $i_2=i_0-i$, where i is the control current, and linearizing the force-current relation, yield the following well-known linear model of the system [6], [7]:

$$m\ddot{y} - c_y y = c_i i \tag{5}$$

where c_y is the position "negative" stiffness, and c_i the current stiffness of the system. Note that since the plant poles are $s_1 = +k$, $s_2 = -k$, where $k = \sqrt{c_y / m}$, the open-loop system (5) is evidently unstable. Applying to system (5) the LQ-design procedure (1)-(4) yields the optimal Proportional-Derivative (PD) control law (see also the Appendix A and [8])

$$i^0 = -(g_1 y + g_2 \dot{y}) \tag{6}$$

where the feedback gains are given by

$$g_1 = mk^2[1 + (1 + 1/\rho k^4)^{1/2}]/c_i\,,$$

$$g_2 = \sqrt{2}mk\ \ [1 + (1 + 1/\rho k^4)^{1/2}]^{1/2} / c_i \tag{7}$$

In order to avoid using the weighting parameter ρ in (7), we reduce the characteristic polynomial of the optimal closed-loop system (5) and (6), $\varphi(s)=s^2+(g_2c_i/m)s+(g_1c_i/m)-k^2$, to the form $\varphi(s)=s^2+2\zeta\omega_0 s+\omega_0^2$, where ω_0 is the desired value of the undamped natural frequency of the system, and ζ, the relative damping. Equating these polynomials, we have $\rho=1/(\omega_0^4-k^4)$, $\omega_0 \geq k$, $\zeta=\sqrt{2(1+k^2/\omega_0^2)}/2$, and finally

$$g_1=m(\omega_0^2+k^2)/c_i\,,\quad g_2=m\sqrt{2(\omega_0^2+k^2)}/c_i \tag{8}$$

Thus, (6) and (8) are the solution to the LQ-design problem for 1-DOF AMB system (5). It contains the varying parameter ω_0 (instead of ρ); ω_0 has a more clear physical sense. The analysis of the dynamic properties of the optimal closed-loop system (5), (6) and (8) is beyond the scope of this paper. Note that the limitating cases $\rho\to 0$ ("cheap" control) and $\rho\to\infty$ ("expensive" control) are analyzed in [6].

III. Five-DOF AMB System

As shown in Fig. 2, a rigid gyroscopic rotor of mass M ,with equatorial and axial principal moments of inertia J_1 and J_3 respectively, spins at the constant rotational speed ω in two radial and one axial AMBs. We shall determine the position of the rotor-attached frame $C\xi\eta\zeta$ with respect to the fixed frame $Oxyz$ by the Cartesian coordinates x_c, y_c, z_c of the origin (center of mass) C and by the angles of tilting φ_x and φ_y about x and y axis, respectively. The vector $q=(q_1, \ldots, q_5)^T=(x_c, y_c, z_c, \varphi_x, \varphi_y)^T$ denotes the output variables of the suspension system.

Let the AMBs contain ten electromagnets having the currents $i=(i_1, \ldots, i_{10})^T$; input voltages $v=(v_1,\ldots, v_{10})^T$; resistances r_s; and inductances $L_{sn}= L_{sn}(q)$; $s,n=1, \ldots, 10$. We introduce the vector of the generalized magnetic forces $F=(F_1, \ldots, F_5)^T$ and the vector of the bearing magnetic forces (or reactions) $Q=(Q_1, \ldots, Q_5)^T$ applied at the journal centres O_1 and O_2, as shown in Fig. 2. The forces Q_1 and Q_3 act in the x-direction, Q_2 and Q_4 act in the y-direction (they are not shown in Fig. 2), and Q_5 acts in the z-direction. The relation between F and Q is given in the Appendix B.

We write the Lagrange-Maxwell dynamic equations in the form

$$M\ddot{x}_c=F_1\,,\quad M\ddot{y}_c=F_1\,,\quad M\ddot{z}_c=F_3 \tag{9}$$

$$J_1\ddot{\varphi}_x+J_3\omega\dot{\varphi}_y=F_4\,,\quad J_1\ddot{\varphi}_y-J_3\omega\dot{\varphi}_x=F_5 \tag{10}$$

$$\sum_{n=1}^{10}L_{sn}\frac{di_n}{dt}+\sum_{n=1}^{10}\sum_{j=1}^{5}\frac{\partial L_{sn}}{\partial q_j}\dot{q}_j i_n+r_s i_s=v_s\,,\quad s=1,\ldots,10 \tag{11}$$

Note that (9) describes the translational motions of the rotor, and that (10), which is coupled by the gyroscopic terms, describes the tilting motions. Equation (11) represents the balance of EMFs and voltages.

The control goal is to stabilize the rotor at the reference position q=0 so that minimization of the perfomance index (2) (where u now denotes the magnetic forces F) and the copper losses in the AMB coils is provided. Decomposing the suspension system (9)-(11) into the mechanical subsystem (9), (10) and the electromagnetic subsystem (11), we shall solve the control problem in two stages. First, we shall find the optimal control forces $F_1=F_1^0$, $F_2=F_2^0$, $F_3=F_3^0$ and the optimal control moments $F_4=F_4^0$ and $F_5=F_5^0$ which minimize the perfomance index (2). Second, we shall determine the optimal control currents $i=i^0$ and voltages $v=v^0$ which produce the optimal forces and moments F^0 and, at the same time, minimize the copper losses in the coils.

There are four control systems at the first stage: three 1-DOF systems (9) and one 2-DOF system (10). By applying the LQ-design procedure (1)-(4) to each of the systems (9), one easily determines that the optimal control forces are given by $F_j^0 = -M(\omega_0^2 q_j + 2\varsigma\omega_0 \dot{q}_j)$, $j = 1,2,3$, where $\omega_0=\rho^{-1/4}$ is the desired undamped natural frequency of the translational motions, and $\varsigma = \sqrt{2}/2$ is the optimal relative damping.

Applying now the LQ-design procedure (1)-(4) to the 2-DOF system (10), one can verify that the fourth-order real (or the second-order complex) Riccati Eq. (4) has an analytical solution (see the Appendix C). Using this property of the system (10), we can formulate a closed-form solution of the LQ-design problem. The optimal control moments are given by

$$\begin{aligned} F_4^0 &= -J_1(k_1\varphi_x + k_2\dot{\varphi}_x + k_3\varphi_y) \\ F_5^0 &= -J_1(k_1\varphi_y + k_2\dot{\varphi}_y - k_3\varphi_x) \end{aligned} \qquad (12)$$

where k_1, k_2 and k_3 are, respectively, the optimal stiffness, damping and radial correcting factors of the tilting motions of the rotor. These factors are given by

$$\begin{aligned} k_1 &= \sqrt{h^4/16 + \Omega_0^4} - h^2/4, \\ k_2 &= \sqrt{2k_1}, \\ k_3 &= h\sqrt{k_1/2} \end{aligned} \qquad (13)$$

Here $h=\omega J_3/J_1$ is the gyroscopic parameter, and $\Omega_0=\rho^{-1/4}$ is the desired undamped natural frequency of tilting motions (about x and y axes) for the non-spinning rotor (i.e. with ω and h equal to zero). The variation of k_1, k_2 and k_3 with the rotational speed ω is shown in Fig.3. As $\omega \to \infty$ the stiffness and damping factors k_1 and k_2 approach to zero, and the radial correcting factor k_3 becomes equal to $k_1(0)$, or Ω_0^2. It is evident, then, that the optimal feedback gains are not constant; they vary with the rotational speed ω in accordance with (13). Note that an optimal speed-dependent controller for AMB is discussed by several authors (for example [6], [9]), but all of them use numerical approaches.

Let us consider now the second stage of the control design problem. The goal is to find the currents $i=i^0$ and voltages $v=v^0$ which produce the optimal generalized forces

$F=F^0$ or the optimal bearing forces $Q=Q^0$ related to F^0 by the linear transformation (see the Appendix B) and, at the same time, minimize the copper losses in the coils. For example, consider use of the actuator shown in Fig.4 to realize bearing force Q_1^0. Let the inductances be given by $L_{11} = k_L/(\delta - k_p x_1)$, $L_{12} = 0$, $L_{22} = k_L/(\delta + k_p x_1)$, where $x_1=x_c+z_1\varphi_y$ is the coordinate of the journal center O_1, and k_L and k_p are the structure parameters of a radial AMB [5] ($k_p = 0.924$ for an eight-pole radial AMB). Accordingly the bearing force is $Q_1 = \left[(\delta - k_p x_1)^{-2} i_1^2 - (\delta + k_p x_1)^{-2} i_2^2\right] k_L k_p / 2$. Optimal currents i_1^0 and i_2^0 must produce the force $Q_1=Q_1^0$. To obtain a unique solution, we introduce the additional relation,

$$r\left[i_1^2(t) + i_2^2(t)\right] = \min, \tag{14}$$

which minimizes copper losses in the coils. The solution is given by

$$\begin{aligned} &i_1^0 = (\delta - k_p x_1)\sqrt{2Q_1^0/k_L k_p}\,, \quad i_2^0 = 0 \quad \text{for } Q_1^0 \geq 0 \\ &i_1^0 = 0, \quad i_2^0 = (\delta + k_p x_1)\sqrt{2\left|Q_1^0\right|/k_L k_p} \quad \text{for } Q_1^0 < 0 \end{aligned} \tag{15}$$

The physical sense of algorithm (15) is obvious : depending on the sign of the force Q_1^0, it is only the first or only the second electromagnet that operates. Note that such a driving mode is known as the external linearization [6].

The voltages $v=v^0$ which induce the optimal currents $i=i^0$ may be found from (11) by designing ten current tracking systems to reduce the tracking errors $\Delta i_s = i_s^0 - i_s$ to zero. For example, the equation in terms of current i_1 can be written, using the assumption $\left|k_p x_1(t)\right| << \delta$, as

$$L_0 \frac{di_1}{dt} + C_v \dot{x}_1 i_1 + r i_1 = v_1 \tag{16}$$

where $L_0 = k_L/\delta$, $C_v = L_0 \cdot k_p/\delta$. Consider the control law given by

$$v_1^0 = \lambda(i_1^0 - i_1) + (C_v \dot{x}_1 + r) i_1 \tag{17}$$

where λ is a positive constant. Substituting (17) into (16) yields the differential equation of the closed-loop tracking system

$$\tau \frac{di_1}{dt} + i_1 = i_1^0 \tag{18}$$

where $\tau = L_0/\lambda$ is the time constant of the system. Thus, the optimal controller obtained due to (15) has a variable structure.

IV. Analysis Of Stability

For simplicity, we analyze stability of the 1-DOF closed-loop system with controlled coordinate x_1 and the actuator shown in Fig.4. Assuming that $\left| k_p x_1(t) \right| << \delta$ and taking (15) and (18) into account, the dynamic equations become

$$m\ddot{x}_1 = Q_1(i_1, i_2) \quad (19)$$

$$Q_1(i_1, i_2) = (i_1^2 - i_2^2) k_L k_p / 2\delta^2 \quad (20)$$

$$\tau \frac{di_1}{dt} + i_1 = i_1^0 , \qquad \tau \frac{di_2}{dt} + i_2 = i_2^0 \quad (21)$$

$$\begin{aligned} i_1^0 &= \delta \sqrt{2Q_1^0 / k_L k_p} , \quad i_2^0 = 0 \quad \text{for } Q_1^0 \geq 0 \\ i_1^0 &= 0, \quad i_2^0 = \delta \sqrt{2\left|Q_1^0\right| / k_L k_p} \quad \text{for } Q_1^0 < 0 \end{aligned} \quad (22)$$

$$Q_1^0 = -m(\omega_0^2 x_1 + 2\zeta\omega_0 \dot{x}_1) \quad (23)$$

where $m = (J_1 + Mz_2^2)/l^2$ is the suspended mass. This is a variable structure system.

We solve the problem of equilibrium state stability of the system (19)-(23) by finding conditions at which periodic motions may occur in the system. Applying the Van der Pol method, we rewrite (19) in the form

$$\ddot{x}_1 + \omega_p^2 x_1 = \psi(i_1, i_2, x_1) \quad (24)$$

where ω_p is the unknown frequency of the periodic motion, and function $\psi(i_1,i_2,x_1) = \left[Q_1(i_1,i_2)/m\right] + \omega_p^2 x_1$ is assumed to be small. The solution of (24) is presented in the form $x_1(t) = a\cos\omega_p t + b\sin\omega_p t$, where a and b are the unknown slowly varying functions of time which satisfy the equations

$$\begin{aligned} \frac{da}{dt} &= -\frac{1}{2\pi}\int_0^\theta \psi(i_1, i_2, x_1)\sin\omega_p t\, dt \\ \frac{db}{dt} &= \frac{1}{2\pi}\int_0^\theta \psi(i_1, i_2, x_1)\cos\omega_p t\, dt \end{aligned} \quad (25)$$

and where $\theta = 2\pi/\omega_p$ is the period of the motion. To solve Eqs.(25), we successively determine the force $Q_1^0(t)$ from (23), the currents $i_1^0(t)$ and $i_2^0(t)$ from (22), and the actual currents $i_1(t)$ and $i_2(t)$ by integrating (21); finally, we substitute the expressions $i_1(t)$ and $i_2(t)$ into (20). Note that, as shown in Fig. 5, the periodic motions may be represented by the elliptic trajectory in the phase plane $x_1, \dot{x}_1$. The straight line $\dot{x}_1 = -(\omega_0 / 2\zeta)x_1$ is the switching line of the electromagnets. At point 1 (t=0, $\theta, 2\theta, ...$) the first electromagnet is switched off and the second electromagnet is switched on; and at point 2 ($t=\theta/2, 3\theta/2, ...$), vice versa. After all conversions we obtain $dA/dt = \mu(\omega_p)/(\tau - 2\zeta/\omega_0)$, where $A = (a^2 + b^2)^{1/2}$ is the amplitude of the periodic motion, and $\mu(\omega_p)$ is a positive function of ω_p. The system considered will be stable if $dA/dt<0$, and unstable if $dA/dt>0$. The stability condition is therefore

$$\tau < 2\zeta / \omega_0 \quad (26)$$

Note that the system instability results from the phase lag of the actual force $Q_1(t)$, and that τ is equal to the time of the system transition from point 2 into point 3 (Fig. 5).

V. Numerical And Experimental Results

To illustrate the validity of the stability condition a cryogenic turboexpander [7] is used here as an example. We have the following parameters: m=2.3 kg, δ=0.3 mm, k_p=0.924, k_L=11.5·10^{-6} Hm. Numerical simulation has been conducted by integrating (19)-(23) with selection of ω_0=500 rad/s (80 Hz) and ζ=0.707 for different time constants τ of the current tracking system. In accordance with the theoretical result, the system is unstable for $\tau>\tau^*$, where $\tau^*=2\zeta/\omega_0$=2.8 ms. Responses of the system to a step of 0.1δ in the displacement x_1 are shown in Fig. 6. It can be seen that the system is stable for $\tau<\tau^*$ (curve 1) but unstable for $\tau>\tau^*$ (curve 3), and that it possesses periodic motion for $\tau=\tau^*$ (curve 2). This numerical result is predicted by the theoretical one.

Figure 7 presents a comparison of the experimental and theoretical diagrams of the system stability in the plane ζ, ω_0 for τ=1.1 ms. The experimental curve deviates from the theoretical straight line due to the influence of the time constant of the differentiator (0.2 ms) that is used to obtain the velocity $\dot{x}_1$, but which is not taken into account in Sec.IV. Therefore, we can say that the theoretical results are qualitatively verified experimentally.

VI. Controller Simplification

Generally speaking, the control system obtained is not very simple to implement because the controller used is switching, multi-coupled, and dependent on the rotational speed ω. Since there is a practical interest in simplification of the controller, let us discuss this problem.

First of all, the simplification may be achieved by using a linear (not switching) controller. A linear control is based on a linearized model of a system. Since a magnetic force is a parabolic function of current, the linearization is impossible at the point with zero current. We must, therefore, introduce bias currents resulting in additional copper losses. Therefore, a linear controller does not minimize copper losses in coils.

The simplification may be also achieved by dividing the large system into subsystems with a local control of each subsystem. Such an approach is called decentralized control and has been discussed by several authors. Schweitzer [6], starting from the fact that the gyroscopic coupling does not destabilize the system, proposes to use a decoupled controller designed for the non-rotating case and claims that the deterioration in perfomance is almost negligible in many cases. Some different results have been obtained in [10]. Figure 8, taken from [10], shows the unbalance responses of the radial AMB with a 1 μm eccentricity of the center of mass of the gyroscopic rotor for the flywheel energy storage system prototype. The nominal rotational speed is 12000 r.p.m. It is easily seen that the centralized controller provides significantly smaller control voltages than the decentralized one (control forces and currents are approximately proportional to voltages). It follows that the most simple decentralized controller can not provide good perfomance in all cases.

VII. CONCLUSION

The paper presents the closed-form solution to the LQ-design problem for two AMB control systems. The result seems to be useful for both high speed AMB applications and pedagogical aims. The minimization of copper losses in coils has yielded a control system with a variable structure. The analysis of stability of the system has resulted in a simple relationship between a natural frequency, damping factor and time constant of the current tracking system. There is a good correlation between analytical, numerical and experimental results. It has been also established that the simplification of the controller by its decentralization may result in a significant deterioration of the system perfomance.

ACKNOWLEDGMENT

The author would like to thank Mikhail Afanasiev and Erkki Lantto for their contribution to the simulations and experiments, and Mike Gorman and George Verghese for helpful discussions.

APPENDIX

A. We reduce (5) to the canonical form (1) and write matrix P as follows:

$$x=(y,\dot{y})^T,\ q=y,\ u=ic_i/m,\quad C=(1,\ 0),$$
$$A=\begin{bmatrix}0 & 1\\ k^2 & 0\end{bmatrix},\quad B=\begin{bmatrix}0\\ 1\end{bmatrix},\quad P=\begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix}$$

Matrix Riccati Eq. (4) yields the set of equations $p_2^2-2\rho\, k^2 p_2-\rho=0$, $p_3^2=2\rho\, p_2$, $p_1=p_2 p_3/\rho-k^2 p_3$, having the evident analytical solution $p_3=\sqrt{2}\,\rho\, k\left[1+(1+1/\rho\, k^4)^{1/2}\right]^{1/2}$,

$$p_2=\rho\, k^2\left[1+(1+1/\rho\, k^4)^{1/2}\right],$$

$$p_1=\sqrt{2}\rho\, k^3(1+1/\rho\, k^4)^{1/2}\left[1+(1+1/\rho\, k^4)^{1/2}\right]^{1/2}.$$

B. Vectors F and Q are correlated by the linear transformation

$$F=ZQ,\qquad Z=\begin{bmatrix}1 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & -z_1 & 0 & -z_2 & 0\\ z_1 & 0 & z_2 & 0 & 0\end{bmatrix}$$

C. Considering the 2-DOF system, we rewrite Eqs.(10) in the complex form

$$\ddot{\varphi}-jh\dot{\varphi}=u$$

where $\varphi = \varphi_x + j\varphi_y$, $u = F_4 / J_1 + jF_5 / J_1$, $h = \omega J_3 / J_1$, $j = \sqrt{-1}$. Using the complex state-space approach, we have

$$x = (\varphi, \dot{\varphi})^T, \quad q = \varphi, \quad B = (0,1)^T, \quad C = (1,0),$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & jh \end{bmatrix}, \quad P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + j \begin{bmatrix} 0 & p_4 \\ -p_4 & 0 \end{bmatrix}$$

The second-order complex matrix Riccati Eq.(4) embodies four scalar equations

$$\begin{aligned} & p_2^2 + p_4^2 - \rho = 0 \\ & p_2 p_3 - \rho\, p_1 + h\rho\, p_4 = 0 \\ & -p_3 p_4 + h\rho\, p_2 = 0 \\ & p_3^2 - 2\rho\, p_2 = 0 \end{aligned}$$

having the analytical solution

$$p_1 = \frac{4}{\rho^2 h^3} p_4^3 + h p_4, \quad p_2 = \frac{2}{\rho\, h^2} p_4^2, \quad p_3 = \frac{2}{h} p_4,$$

$$p_4 = \left[\left(\frac{\rho^4 h^8}{64} + \frac{\rho^3 h^4}{4} \right)^{1/2} - \frac{\rho^2 h^4}{8} \right]^{1/2}$$

The same result can be obtained by using the real state-space approach. In this case we write

$$x = (\varphi_x, \dot{\varphi}_x, \varphi_y, \dot{\varphi}_y)^T, \quad q = (\varphi_x, \varphi_y)^T,$$
$$u = (F_4/J_1, F_5/J_1)^T$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -h \\ 0 & 0 & 0 & 1 \\ 0 & h & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \\ \pi_2 & \pi_5 & \pi_6 & \pi_7 \\ \pi_3 & \pi_6 & \pi_8 & \pi_9 \\ \pi_4 & \pi_7 & \pi_9 & \pi_{10} \end{bmatrix}$$

The fourth-order real matrix Riccati Eq.(4) embodies ten scalar equations (for brevity, they are omitted). One can verify that because of the symmetries of the model (10) the following six relations take place: $\pi_3 = \pi_7 = 0, \pi_1 = \pi_8, \pi_2 = \pi_9, \pi_4 = -\pi_6, \pi_5 = \pi_{10}$ (these relations can be obtained, for example, by expanding the functions $\pi_s(h), s = 1,...,10$, into a power series). For this reason, the Riccati Eq.(4) actually only embodies the four scalar equations obtained above (where $p_1=\pi_1$, $p_2=\pi_2$, $p_3=\pi_5$, $p_4=\pi_6$). The advantages of the complex state-space approach are evident.

References

[1] A.I.Lurie, *Some Non-Linear Problems of Control Theory*. Gostechisdat, Moscow, 1951 (in Russian).

[2] R.E.Kalman, "Contributions to the theory of optimal control", *Bol. Soc. Mat. Mexicana*, 5, pp.102-119, 1960.

[3] H.Kwakernaak and R.Sivan, *Linear Optimal Control Systems*. Wiley-Interscience, 1972.

[4] W.Leonard, *Control of Electrical Drives*. Springer-Verlag, 1985.

[5] C.S. Kim and C.W. Lee, "Isotropic optimal control of active magnetic bearing system", in *Proc. 4th Int. Symp. on Magnetic Bearings* (Zurich, Switzerland), pp. 35-40, Hochschulverlag AG an der ETH Zurich, 1994.

[6] G.Schweitzer, H.Bleuler and A.Traxler, *Active Magnetic Bearings*. Hochschulverlag AG an der ETH Zurich, 1994.

[7] Y.N.Zhuravlyov, *Active Magnetic Bearings*. Report 37, Helsinki Univ. of Tech., 1992.

[8] R.D.Hampton, C.M.Grodsinsky, P.E.Allaire, D.W.Lewis and C.R.Knospe, "Microgravity vibration isolation: An algebraic introduction", *Journal of the Astronautical Sciences*, Vol.40, No.2, pp.241-259, April-June 1992.

[9] R.Larsonneur, "A high speed rotor in active magnetic bearings", in *Proc. 3rd Int. Conf. on Rotordynamics*, IfToMM, Lyon, France, Sept. 1989.

[10] Y.N.Zhuravlyov, M.V.Afanasiev and E.Lantto, "Inverse problems of magnetic bearing dynamics", in *Proc.4th Int. Symposium on Magnetic Bearings* (Zurich, Switzerland), pp. 79-84, Hochschulverlag AG an der ETH Zurich, 1994.

FIGURE CAPTIONS

For the paper No. 96-032 by Yuri N. Zhuravlyov “On LQ-control of magnetic bearing”

Fig. 1. Model of 1-DOF magnetic bearing system.

Fig. 2. Model of rigid gyroscopic rotor-magnetic bearing system.

Fig. 3. Variation of optimal stiffness (k_1), damping (k_2) and radial correcting (k_3) factors with rotational speed.

Fig. 4. Radial bearing actuator generating force Q_1.

Fig. 5. Periodic motion trajectory in the phase plane.

Fig. 6. Step responses of suspension system: (1) τ=1.4 ms, (2) τ=2.8 ms, (3) τ=5.6 ms.

Fig. 7. Theoretical (1) and experimental (2) stability boundaries for τ=1.1 ms .

Fig. 8. Unbalance responses: (1) centralized control, (2) decentralized control.